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# RINGS RADICAL OVER SUBRINGS

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ABSTRACT

Let $R$ be a ring with a subring $A$ such that a power of every element of $R$ lies in $A$. The following results are proved: If $R$ has no nonzero nil right ideals, neither does $A$; if moreover $R$ is prime, $A$ is also prime. If $R$ is semiprime Goldie, so is $A$. If $A$ has no nonzero nilpotent elements, then the nilpotent elements of $R$ form an ideal. Finally if $R$ has no nil right ideals and $A$ is Goldie, then $R$ is Goldie.

In what follows $R$ will denote an associative ring, not necessarily with 1, and $A$ will always denote a subring of $R$. Following Faith [1], we say $R$ is $A$*-radical* if for each $r \in R$ there exists $n = n(r) \geqq 1$ such that $r^n \in A$. In this paper we study the relationship between the properties of $R$ and $A$ when $R$ is $A$-radical. The first section deals with some results of independent interest. In Section 2 we place conditions on $R$ and show that the same conditions are forced on $A$. Finally, in Section 3 we show that by placing conditions on $A$ these are forced on $R$ provided $R$ is without nonzero nil ideals (except Theorem 7 where we require $R$ to be without nonzero nil right ideals). For a good cross-section of the results obtained in this kind of study one can look in [1], [4], [6] and [7].

The following fact will be used without further mention: if $R$ is $A$-radical and $r_1, \cdots, r_m \in R$, then there exists $k = k(r_1, \cdots, r_m) \geqq 1$ such that $r_i^k \in A$, $i = 1, \cdots, m$.

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## 1. Preliminary results

We begin with

LEMMA 1. *Let $R$ be a ring with no nonzero nil right ideals. Suppose $a \in R$ and $ax^{n(x)}a = 0$, $n(x) \geqq 1$, all $x \in R$. Then $a = 0$.*

PROOF. Clearly $a$ is nilpotent. If $a^2 \neq 0$ and $k$ is minimal such that $a^k = 0$, then $a^{k-1}$ satisfies the same hypothesis as $a$ and $(a^{k-1})^2 = 0$. Hence we may assume $a^2 = 0$. Let $r \in R$ with $r^2 = 0$; we claim that $ara = 0$. In fact, if $x \in R$ there exists $n \geqq 1$ such that $a((axar) + r)^n a = 0$. Since $r^2 = a^2 = 0$, if $n = 1$ we get $ara = 0$; if $n > 1$, then $ar(axar)^{n-1}a = 0$. In any case we have $(arax)^n = 0$. Thus $araR$ is a nil right ideal and consequently, from the hypothesis on $R$, $ara = 0$.

Let $r, s \in R$ with $rs = 0$. Then $(sxr)^2 = 0$ for every $x \in R$ and so, by the above, $a(sxr)a = 0$ for every $x \in R$. Thus, since $R$ is semiprime, $ras = 0$.

Let $x \in R$ and let $n \geqq 1$ such that $ax^n a = 0$; we show that $(ax)^{n+1} = 0$. If $n = 1$ this is clear. If $n > 1$, $(ax)(x^{n-1}a) = ax^n a = 0$ and by what we deduced before this implies $(ax)a(x^{n-1}a) = 0$; continuing in this way we obtain $(ax)^n a = 0$ and so $(ax)^{n+1} = 0$. In other words, $aR$ is a nil right ideal. Therefore, by our hypothesis on $R$, we must have $a = 0$.

THEOREM 1. *Let $R$ be a prime ring with no nonzero nil right ideals. Suppose $a, b \in R$ and $ax^{n(x)}b = 0$, $n(x) \geqq 1$, all $x \in R$. Then $a = 0$ or $b = 0$.*

PROOF. Assume $b \neq 0$ and fix for each $x \in R$ an integer $n(x) \geqq 1$ such that $ax^{n(x)}b = 0$. Then $\rho = \{y \in R \mid ax^{n(x)}y = 0, \text{ all } x \in R\}$ is a nonzero right ideal of $R$ and by Lemma 1, $\rho a = 0$. Thus, since $R$ is prime and $\rho \neq 0$, $a = 0$.

Since prime (nontrivial) nil rings exist, Theorem 1 does not remain valid if one just assumes $R$ to be a prime ring; we believe however that the result remains valid if one replaces the assumption "with no nonzero nil right ideals" by its two-sided version "with no nonzero nil ideals".

REMARK 1. Let $R$ be a prime ring with nontrivial center. Suppose $a, b \in R$ and $ax^{n(x)}b = 0$, $n(x) \geqq 1$, all $x \in R$. If char $R = 0$ or char $R = p \neq 0$ where $p \nmid n(x)$, all $x \in R$, then $a = 0$ or $b = 0$.

PROOF. Since a nonzero element in the center of a prime ring is not a zero divisor we clearly have $ab = 0$. Let $r \in R$ with $r^2 = 0$. If $c \neq 0$ is a central element of $R$ there exists $n \geqq 1$ such that $a(c + r)^n b = 0$ where if char $R = p \neq 0$, $p \nmid n$. Since $r^2 = 0$ and $c$ is not a zero divisor it follows that $arb = 0$.

Let $r,\ s \in R$ with $rs = 0$. Then, since $(sxr)^2 = 0$ for every $x \in R$, we get $asRrb = 0$. But $R$ is prime so $as = 0$ or $rb = 0$.

Now if $x \in R$, $ax^n b = 0$ for a suitable $n$. Pick $k \geqq 1$ minimal such that $ax^k b = 0$. If $k > 1$, by what we have deduced above, $0 = ax^k b = (ax^{k-1})(xb)$ implies $axb = 0$ or $ax^{k-1} b = 0$. This contradicts the minimal nature of $k$. Thus $aRb = 0$, and, since $R$ is prime, $a = 0$ or $b = 0$.

For later reference we state the well known

THEOREM (Levitzki [3, lemma 1.1]). *Let $R$ be a ring and $0 \neq \rho$ a nil right ideal of $R$. Suppose that given $a \in \rho$, $a^n = 0$ for a fixed integer $n$; then $R$ has a nonzero nilpotent ideal.*

Note that using Levitzki's result we can drop the assumption "with no nonzero nil right ideals" in Theorem 1 if the integers $n(x)$ have a finite maximum as $x$ ranges over $R$.

## 2. Going down

A natural question is: "If $R$ has no nonzero nil ideals and $R$ is $A$-radical, is $A$ without nonzero nil ideals?" The next result gives us an affirmative answer to this question modulo the Koethe conjecture.

THEOREM 2. *If $R$ has no nonzero nil right ideals and $R$ is $A$-radical, then $A$ has no nonzero nil right ideals.*

PROOF. Suppose $\rho \neq 0$ is a nil right ideal of $A$. Let $r \in R$ with $r^2 = 0$. If $a \in \rho$ there exists $n \geqq 2$ such that $(ar)^n$ and $(ar + r)^n$ are in $A$. Thus $r(ar)^{n-1} \in A$ and so $ar(ar)^{n-1} = (ar)^n \in \rho$; hence $ar$ is nilpotent. Since $(rxr)^2 = 0$ for every $x \in R$, we have $rxra$ nilpotent for every $x \in R$. Hence $rarR$ is nil and by hypothesis this implies $rar = 0$. In short, if $r \in R$ and $r^2 = 0$ then $r\rho r = 0$.

By Lemma 1, $A$ is semiprime and so by Levitzki's theorem there exists $a \in \rho$ with $a^k = 0$, $a^{k-1} \neq 0$, and $k \geqq 4$. Now, since $k \geqq 4$, $(a^{k-2})^2 = 0$ and consequently $a^{k-2}\rho\, a^{k-2} = 0$. Thus $a^{k-2}\rho$ is a nilpotent right ideal of $A$. Since $A$ is semiprime we must have $a^{k-2}\rho = 0$ and so $a^{k-1} = 0$, a contradiction.

When $R$ has no nonzero nil ideals and $R$ is $A$-radical, every nonzero ideal of $R$ intersects $A$ nontrivially. So, in this situation it is clear that if $A$ is prime, $R$ is also prime.

Combining Theorems 1 and 2 we have

THEOREM 3. *If $R$ is prime with no nonzero nil right ideals and $R$ is $A$-radical, then $A$ is prime with no nonzero nil right ideals.*

We continue with

LEMMA 2. *Suppose $R$ has no nonzero nil right ideals and $R$ is $A$-radical. If $a_1, a_2 \in A$ and $a_1A \cap a_2A = 0$, then $a_1R \cap a_2R = 0$.*

PROOF. Let $a_1, a_2 \in A$ with $a_1A \cap a_2A = 0$. Consider the right ideal of $A$, $\rho = a_1R \cap a_2A$. Let $x \in \rho$, say $x = a_1 r_1 = a_2 r_2$, $r_1 \in R$ and $r_2 \in A$, and let $n \geqq 1$ such that $(r_1 a_1)^n \in A$. Since $a_1, r_2 \in A$ we have

$$a_1(r_1 a_1)^n = (a_1 r_1)^n a_1 = (a_2 r_2)^n a_1 \in a_1A \cap a_2A = 0;$$

so $x^{n+1} = (a_1 r_1)^{n+1} = 0$. Thus $\rho$ is nil and consequently by Theorem 2, since $R$ has no nonzero nil right ideals, $\rho = a_1R \cap a_2A = 0$.

Let $x \in a_1R \cap a_2R$, say $x = a_1r_1 = a_2r_2$, and let $n \geqq 1$ such that $(r_2 a_2)^n \in A$. Then $(a_1 r_1)^n a_2 = (a_2 r_2)^n a_2 = a_2(r_2 a_2)^n \in a_1 R \cap a_2A = 0$ and so $x^{n+1} = (a_2 r_2)^{n+1} = 0$. Thus $a_1R \cap a_2R$ is nil and by hypothesis we must have $a_1R \cap a_2R = 0$.

Recall that a left ideal $\lambda$ of a ring $R$ is said to be *essential* if it intersects every nonzero left ideal of $R$ nontrivially.

REMARK 2. Suppose $R$ has no nonzero nil right ideals, $R$ is $A$-radical, and $\lambda$ is an essential left ideal of $R$. Then $\lambda \cap A$ is an essential left ideal of $A$.

PROOF. Let $a \neq 0$ in $A$. Then $\lambda \cap Ra \neq 0$ and there exists $b$ in $\lambda \cap Ra$, $b$ not nilpotent. Let $n \geqq 1$ such that $b^n \in A$. Since $b^{n+1} \neq 0$ is in $Rb^n \cap Ra$, by the left analogue of Lemma 2, $Ab^n \cap Aa \neq 0$. In particular $(\lambda \cap A) \cap Aa \neq 0$. This being true for every $a \neq 0$ in $A$, we conclude that $\lambda \cap A$ is an essential left ideal of $A$.

If $S \subset R$ is a subset and $r \in R$ let $l_S(r) = \{x \in S \mid xr = 0\}$, the left annihilator of $r$ in $S$. The *left singular ideal* of $R$ is $Z(R) = \{r \in R \mid l_R(r) \text{ is essential}\}$, seen to be an ideal of $R$. Let $Z(A)$ denote the left singular ideal of the subring $A$ of $R$. Since for $r \in R$, $l_A(r) = l_R(r) \cap A$, we have the

COROLLARY. *If $R$ has no nonzero nil right ideals and $R$ is $A$-radical, then $Z(A) = Z(R) \cap A$.*

In [7] Rowen conjectured that if $R$ is a prime left Goldie ring and $R$ is $A$-radical, then either $R$ is commutative or $R$ and $A$ are left orders in the same simple artinian rings. We conclude this section with a step in this direction. We will follow the arguments of Procesi and Small of Goldie's theorem for semiprime rings [3, chap. 4].

REMARK 3. Suppose $R$ is a semiprime left Goldie ring and $R$ is $A$-radical. If $\lambda$ is an essential left ideal of $A$, then $\lambda$ contains a regular element.

PROOF. Assume first $R$ is a prime left Goldie ring. Since a semiprime left Goldie ring has no nonzero nil right ideals [5], by Theorem 3 $A$ is prime. Also, as a subring of $R$, $A$ inherits the ascending and descending chain conditions on left annihilators. Choose $a \in \lambda$ so that $l_A(a)$ is minimal. Suppose $l_A(a) \neq 0$. Then $Ra$ is not essential in $R$ and so clearly $Aa$ is not essential in $A$. Let $J \neq 0$ be a left ideal of $A$ such that $Aa \cap J = 0$. Since $\lambda$ is essential, $\lambda \cap J \neq 0$ hence we may assume $J \subset \lambda$. If $x \in J$, $r \in l_A(a+x)$, $ra = -rx \in Aa \cap J = 0$ thus $r \in l_A(a) \cap l_A(x)$. By the minimality of $l_A(a)$ we get $l_A(a) \subset l_A(x)$ for all $x \in J$; thus $l_A(a)J = 0$. Since $l_A(a) \neq 0$, $J \neq 0$ and $A$ is prime, this is a contradiction. Hence $l_A(a) = 0$ and since $R$ has no nonzero nil right ideals $l_R(a) = 0$. Thus, since $R$ is prime Goldie, $a$ is regular in $R$.

Now, let $R$ be a semiprime left Goldie ring and let $S_1 \oplus \cdots \oplus S_n$ be a maximal direct sum of minimal annihilator ideals. Each $S_i$ is a prime left Goldie ring [cf. 3, lemma 4.17] $S_i \cap A$-radical and clearly $\lambda \cap S_i$ is an essential left ideal of $S_i \cap A$. Thus, each $\lambda \cap S_i$ contains an element $a_i$ regular in $S_i$. If $a = a_i + \cdots + a_n$, then [cf. 3, lemma 4.18] $a$ is regular in $R$.

THEOREM 4. *If $R$ is a semiprime left Goldie ring and $R$ is $A$-radical, then $A$ is a semiprime left Goldie ring.*

PROOF. By [5] $R$ has no nonzero nil right ideals, so by Lemma 1 $A$ is semiprime. Let $a \in A$ regular and $b \in A$. Then, since $R$ has no nonzero nil right ideals, $a$ is regular in $R$ and consequently $Ra$ is essential in $R$. By Lemma 2 $Aa$ is essential in $A$; as is easy to see, $\lambda = \{x \in A \mid xb \in Aa\}$ is also an essential left ideal of $A$. By the preceding discussion $\lambda$ contains a regular element $c$. Thus $cb = da$, some $d \in A$. Hence $A$ satisfies the left Ore conditions, so has a ring of left quotients $Q(A)$.

Let $I \neq 0$ be a left ideal of $Q(A)$. Then by Zorn's lemma there exists a left ideal $K$ in $A$ such that $(I \cap A) \oplus K$ is essential in $A$. Thus, as before, $(I \cap A) \oplus K$ contains a regular element, therefore $Q(A) = I \oplus Q(A)K$ and if 1 is the unit element of $Q(A)$, $1 = e + f$ where $e \in I$, $f \in Q(A)K$. It is easy to see that $I = Ie = Q(A)e$. Thus every nonzero left ideal of $Q(A)$ is principal, generated by an idempotent. Thus $Q(A)$ is semiprime and every left ideal of $Q(A)$ is a left annihilator.

Now, $R$ is a left order in a semisimple (left) artinian ring $Q(R)$. By universality we may assume $Q(A) \subset Q(R)$. Thus $Q(A)$ satisfies the descending chain condition of left annihilators, so on all left ideals. Since $Q(A)$ is semiprime, $Q(A)$ is semisimple (left) artinian. Therefore $A$ is semiprime (left) Goldie.

## 3. Going up

The *hypercenter,* $T$, of a ring $R$ is defined by $T = \{t \in R \mid tx^n = x^n t,\ n = n(xt) \geqq 1, \text{all } x \in R\}$. Herstein [4] has shown that if $R$ is a ring with no nonzero nil ideals then $T = Z$, the center of $R$. This result enables us to avoid the Köethe conjecture in Theorem 6. Right now we use it in the following

THEOREM 5. *Suppose $R$ is 2-torsion free and is $A$-radical. If $A$ has no nonzero nil right ideals, then, $N(R)$, the upper nil radical of $R$, contains every nil one-sided ideal of $R$.*

PROOF. Since $R/N(R)$ is radical over $A + N(R)/A \cong A/A \cap N(R) \cong A$, we may assume $N(R) = 0$, and, we have to show $R$ has no nonzero nil one-sided ideals.

Suppose $\rho \neq 0$ is a nil right ideal of $R$. Since $R$ is semiprime, by Levitzki's theorem there exists $x \in \rho$ with $x^k = 0$, $x^{k-1} \neq 0$, and $k \geqq 6$. For $k$ odd, $3(k-1)/2 \geqq k$, and $(x^{(k-1)/2})^3 = 0$; for $k$ even, $3(k-2)/2 \geqq k$, and $(x^{(k-2)/2})^3 = 0$. Thus we may assume $x^3 = 0$ and $x^2 \neq 0$. If $r \in R$, we can find an integer $n$ such that

(1) $r^n \in A$

(2) $(1+x)r^n(1-x+x^2) = ((1+x)r(1-x+x^2))^n \in A$

(3) $(1-x)r^n(1+x+x^2) \in A$

(4) $(1-x+x^2)r^n(1+x) \in A$

(5) $(1+x+x^2)r^n(1-x) \in A$.

Now, from (1), (2), and (3), we get $2(-xr^n x + r^n x^2) \in A \cap Rx = 0$; similarly (1), (4), and (5) give us $2(-xr^n x + x^2 r^n) \in A \cap xR = 0$ (for if $A$ has no nonzero nil right ideals, then, clearly, it has no nonzero nil left ideals). By hypothesis we must have $x^2 r^n = xr^n x = r^n x^2$. In other words, $x^2 \in$ hypercenter of $R$. Since $N(R) = 0$, $x^2$ is a central element. But the center of a semiprime ring has no nonzero nilpotent elements, so $x^2 = 0$, a contradiction.

We believe the above result is true with no assumptions on torsion.

THEOREM 6. *Suppose $R$ has no nonzero nil ideals and $R$ is $A$-radical. Then if $A$ has no nonzero nilpotent elements, $R$ has no nonzero nilpotent elements.*

PROOF. We begin by showing $R$ has no nonzero nil right ideals. Assume $\rho \neq 0$ is a nil right ideal of $R$. Let $a \neq 0$ in $\rho$ with $a^2 = 0$. If $r \in R$, then, for some $n$, $r^n$ and $((1+a)r(1-a))^n$ are in $A$. But $((1+a)r(1-a))^n = (1+a)r^n(1-a) = r^n + ar^n - r^n a - ar^n a$, and so $b = ar^n - r^n a - ar^n a \in A$. Now, $b^2 = (ar^n)^2 + (r^n a)^2 - ar^{2n}a$, and by induction we get $b^{2^t} = (ar^n)^{2^t} + (r^n a)^{2^t} + ac_t a$ for some $c_t \in R$. Since $a \in \rho$, $ar^n$ and $r^n a$ are nilpotent, so there exists $t_0$ such that $(ar^n)^{2^{t_0}} = (r^n a)^{2^{t_0}} = 0$; thus $b^{2^{t_0+1}} = (ac_{t_0}a)^2 = 0$, and, from the hypothesis on $A$, we get $b = 0$. Thus $0 = ba = ar^n a$, so $0 = b = ar^n - r^n a$. It follows that $a \in$ hypercenter of $R$. Since $R$ has no nonzero nil ideals, $a$ is a central element. But $R$ is semiprime and $a$ is nilpotent so we must have $a = 0$, a contradiction.

Assume now $a \in R$ with $a^2 = 0$. If $r \in R$, $(ar)^n$ and $((1+a)(ar)(1-a))^n$ are in $A$ for a suitable $n$. But $((1+a)(ar)(1-a))^n = (1+a)(ar)^n(1-a) = (ar)^n - (ar)^n a$, so $(ar)^n a = b \in A$. Since $b^2 = 0$ and $A$ has no nonzero nilpotent elements, $b = 0$. Hence $(ar)^{n+1} = 0$. This shows that $aR$ is nil right ideal and consequently $a = 0$. Therefore $R$ has no nonzero nilpotent elements.

An immediate consequence, which is in fact equivalent to Theorem 6 is: "If $R$ is $A$-radical and $A$ has no nonzero nilpotent elements, then the nilpotent elements of $R$ form an ideal" (Proof: Consider $N(R)$ = upper nil radical of $R$ and $\bar{R} = R/N(R)$. $\bar{R}$ is $\bar{A}$-radical, where $\bar{A} = A + N(R)/N(R) \cong A/A \cap N(R) \cong A$ has no nonzero nilpotent elements. Thus $\bar{R}$ has no nonzero nilpotent elements, which is to say that every nilpotent element of $R$ lies in $N(R)$, i.e., $N(R) = \{$nilpotent elements of $R\}$.)

COROLLARY (Rowen). *Suppose $R$ has no nonzero nil ideals, $R$ is $A$-radical, and $A$ is a domain. Then $R$ is a domain.*

PROOF. By an earlier observation, under these hypothesis $R$ is a prime. Moreover, by Theorem 6, $R$ has no nonzero nilpotent elements. It follows that $R$ is a domain.

Goldie [2] characterized left Goldie rings as those having left singular ideal zero and not containing infinite direct sums of left ideals. This characterization and the Corollary to Lemma 2 give us our last result.

THEOREM 7. *If $R$ has no nonzero nil right ideals, $R$ is $A$-radical, and $A$ is left Goldie, then $R$ is left Goldie.*

*Added in proof.* Herstein has pointed out to us the following:

REMARK. Suppose $R$ has no nonzero nil ideals and $2R = 0$. If $R$ is $A$-radical and $A$ has no nonzero nil right ideals then $R$ has no nonzero nil right ideals.

In fact, suppose $\rho \neq 0$ nil right ideal of $R$. As in Theorem 5 we can find $x \in \rho$ with $x^3 = 0$ and $x^2 \neq 0$. If $a \in A$ let $n \geqq 1$ such that $(1+x)a^n(1+x+x^2)$, $(1+x+x^2)a^n(1+x)$ and $(1+x^2)a^n(1+x^2)$ are in $A$. Then $a_1 = xa^nx^2 + x^2a^nx + x^2a^nx^2 \in A \cap \rho = 0$; hence $0 = xa_1 = x^2a^nx^2$ and, since $(1+x^2)a^n(1+x^2) \in A$, $x^2a^n + a^nx^2 \in A$.

If $b \in A$ let $c = (x^2a^n + a^nx^2)b$. Then $x^2c = 0$. But proceeding as above we have $x^2c^m + c^mx^2 \in A$ for some $m \geqq 1$. Hence $c^mx^2 \in A \cap Rx = 0$. Now, $c^{m+1} = c^m(x^2a^n + a^nx^2)b = c^ma^nx^2b \in Rx^2b$ so $c^{m+1}$, and consequently $c$, is nilpotent. Thus $(x^2a^n + a^nx^2)A$ is a nil right ideal of $A$. By hypothesis we must have $x^2a^n + a^nx^2 = 0$. It follows that $x^2 \in$ hypercenter of $R$ and since $R$ has no nonzero nil ideals $x^2 \in$ center of $R$. But $R$ is semiprime and $x^2$ is nilpotent so $x^2 = 0$, a contradiction.

We can now sharply improve Theorem 5 by removing the assumption of no 2-torsion.

THEOREM 5'. *If $R$ is $A$-radical and $A$ has no nonzero nil right ideals then $N(R)$, the upper nil radical of $R$, contains every nil one-sided ideal of $R$.*

PROOF. By going to $R/N(R)$ we may assume $N(R) = 0$, and we have to show $R$ has no nonzero nil one-sided ideals. Consider the ideal $U = \{x \in R \mid 2^kx = 0$ for some $k \geqq 1\}$. We have that $R/U$ is $A/A \cap U$-radical, and as it is easy to see $R/U$ is 2-torsion free with no nonzero nil ideals, and $A/A \cap U$ has no nonzero nil right ideals. By Theorem 5 it follows that $R/U$ has no nonzero nil right ideals. Clearly the result now will follow if we show $U$ has no nonzero nil right ideals. But $U$ is $A \cap U$-radical, $2U = 0$ (for $2U$ is a nil ideal of $R$), and $U$ has no nonzero nil ideals; moreover since $A \cap U$ is an ideal of $A$, $A \cap U$ has no nonzero nil right ideals. Therefore, by the preceding remark $U$ has no nonzero nil right ideals.

Note that Theorem 6 follows now immediately. Note also that we have

THEOREM 7'. *If $R$ has no nonzero nil ideals, $R$ is $A$-radical, and $A$ is semiprime left Goldie, then $R$ is left Goldie.*

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